Introduction to Feedback

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This paper gives an introduction to feedback, opamps, and phase-locked loops with an emphasis on demonstrating how one can quickly understand the behavior of simple feedback circuits without detailed calculations by examining the circuit and using high level reasoning.

Contents

1 Introduction

Feedback is used extensively in engineering as a universal way of making things behave. The basic idea is this: you have something that performs a function that you need, but does not do it as well as you would like, so you come up with some way of representing the desired behavior, you then subtract the desired behavior from the actual behavior and take this difference, the error, and feed it back into the input of the function in such a way as to counteract and reduce the imperfection. This is referred to a negative feedback because you choose the sign of your feedback to reduce the response of your system with the idea that you are reducing the undesired behavior more than you are reducing the desired behavior.

You have to be careful of the sign here, otherwise you may create positive feedback. In this case you are taking some of the output and feeding it back into the input in such a way to make the output larger. This can explode on you.

Another thing you need to be careful of is delay around the loop. It may be, if the signal you are processing with your feedback system is oscillatory, that the delay converts your negative feedback to positive feedback. When this occurs your feedback system begin self-sustained oscillations. Engineers study feedback systems in great depth, and a large part of that is so that they can avoid this type of instability, but this question of stability will not be addressed in this paper.

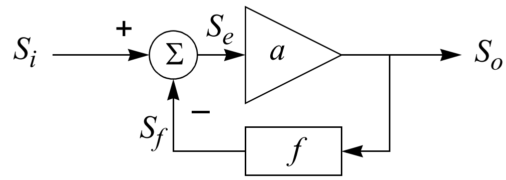
2 An Ideal Feedback System

Figure 1 shows a highly idealized feedback system [1]. In this system the triangle represents the original system (before feedback is applied) and the rectangle represents the feedback. In this case we will assume that the triangle is an amplifier with gain a, thus $S_o = a\, S_e$. Furthermore, f represents gain of the feedback, thus $S_f = f\, S_o$. The little circle that contains a Sigma (Σ) is a summer with two inputs, a non-inverting input marked with a plus sign (+) and the inverting input marked with a minus sign (–), thus $S_e = S_i - S_f$. Combining these relationship allows us to compute the output as a function of the input:

$$S_o = aS_e = a(S_i - S_f) = a(S_i - fS_o) \quad (1)$$

$$S_o = \frac{a}{1 + af} S_i . \quad (2)$$

FIGURE 1 *An idealized feedback system*



It is now helpful to define some standard terms:

a is referred to as the *open-loop gain*,

f is referred to as the *feedback factor*,

$T = af$ is referred to as the *loop gain*, and

$A = \frac{a}{1 + af} = \frac{a}{1 + T}$ is referred to as the *closed-loop gain* of the feedback system.

Thus, a is the gain of the amplifier before feedback is applied, A is the gain after feedback is applied, and T is the gain around the loop. The loop gain T is not directly observable and will not be used further in this paper, but it is of central importance when studying the stability of the loop.

It in interesting to see how this system behaves when the open-loop gain is very large:

$$A_\infty = \lim_{a \to \infty} \frac{a}{1 + af} = \frac{1}{f} . \quad (3)$$

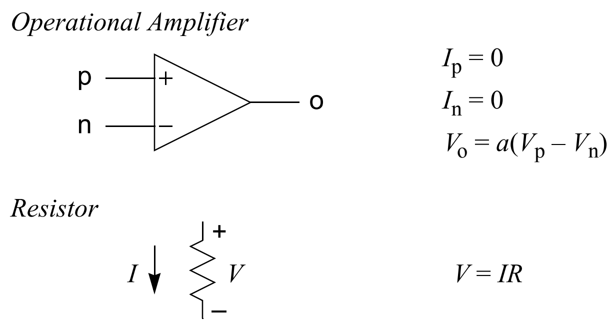
In other words, as the open loop gain of the amplifier goes to infinity, the overall gain of the feedback system becomes $1/f$. Thus to get a overall gain of 2, you would make the feedback factor $f = ½$, which is an attenuator. Attenuators are generally constructed from simple passive components which can be made relatively precisely whereas the amplifier must be made of active components such as transistors and as a result the open-loop gain is often poorly controlled. By simply making the open-loop gain large you can now control the overall gain of the system precisely using a passive attenuator.

Correcting the gain is only one benefit from feedback. It turns out that most of the imperfections exhibited by the amplifier are reduced by a factor of $1 + af$, meaning that the larger the open-loop gain, the better the overall behavior becomes. Two examples include linearity and bandwidth, meaning that both the linearity and bandwidth of the closed-loop feedback system will be better than that of the open-loop amplifier by a factor of $1 + af$. In effect, feedback allows us to trade gain for performance.

3 Non-Inverting Amplifier

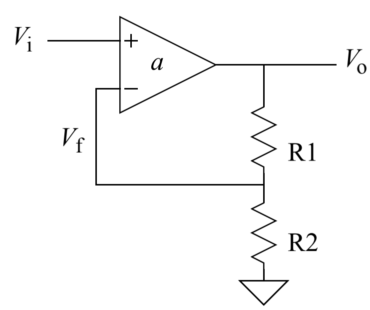
Now consider a practical feedback system constructed with resistors and operational amplifiers (opamps), both of which are shown individually in Figure 2. Here we are assuming a relatively ideal operational amplifier whose input currents are always negligible (approximated as being 0) and whose gain a is always large. Furthermore we assume a linear resistor with resistance R.

FIGURE 2 *The two components used in our feedback amplifier*



Now consider the circuit shown in Figure 3. To derive the equations that describe this circuit we need to first come up with an equation that relates V_f to V_o using the characteristics of the resistor, then we need to calculate V_o from V_i and V_f using the characteristics of the opamp. The resistors are organized into a resistor string connected to ground. The resistor string acts as a voltage divider where the voltage at the output of the voltage divider (V_f) is always a fixed fraction of the voltage at the input of the divider (V_o). The voltage at top of the resistor string is V_o and the voltage at the bottom is 0. This causes a current to flow through the resistor string. The same current flows through both resistors because no current flows into the negative input of the opamp. The current will equal

FIGURE 3 *A non-inverting feedback amplifier*



$$V_o = V_{R1} + V_{R2} = R1I + R2I \tag{4}$$

$$I = \frac{V_o}{R1 + R2}. \tag{5}$$

Then V_f can be found by computing the voltage across R2 that results from this current:

$$V_f = R2I = \frac{R2}{R1 + R2} V_o . \quad (6)$$

Thus, the gain from output to the feedback input is:

$$f = \frac{R2}{R1 + R2} . \quad (7)$$

Using the equation for the opamp allows us to compute the output voltage:

$$V_o = a(V_i - V_f) \quad (8)$$

$$V_o = a\left(V_i - \frac{R2}{R1 + R2} V_o\right) \quad (9)$$

$$V_o = \frac{a}{1 + a\frac{R2}{R1 + R2}} V_i = \frac{a(R1 + R2)}{R1 + R2 + aR2} V_i \quad (10)$$

Now, if we assume that a is very large:

$$V_o = \frac{R1 + R2}{R2} V_i \quad (11)$$

Thus the output voltage has the same sign as the input voltage and is larger in magnitude (assuming the resistor values are positive). Thus, this is a non-inverting amplifier, the gain of which is:

$$A = \frac{R1 + R2}{R2} \quad (12)$$

4 The Virtual Short-Circuit Principle

Consider the input voltage to the opamp in Figure 3 when the open loop-gain a is very large:

$$V_e = V_i - V_f = V_i - \frac{R2}{R1 + R2} V_o = V_i - \frac{R2}{R1 + R2} a V_e = V_i - afV_e \quad (13)$$

$$V_e = \frac{V_i}{1 + af} . \quad (14)$$

This implies that as a becomes large, V_e becomes small. Or more precisely,

$$\lim_{a \to \infty} V_e = 0 . \quad (15)$$

For the rest of this paper, assume that a is very large. Then the voltage at the input of the opamp can assumed to be zero (by this I mean that the voltage difference between the two inputs is zero, not that the voltage of both of the inputs is zero). It is as if the two inputs are shorted. Oddly, we know from the equations that define the opamp, that the current into either input is also zero. Thus, when the open loop gain is large and the feedback is working properly then both the voltage across the input pins and the current into the input pins are virtually zero. How are these two seemingly contradictory facts possible? The feedback always acts to reduce the opamp input voltage. Since the opamp has very high gain, the feedback will be successful at reducing the input voltage essentially to zero as long as the feedback is working. It must, otherwise the output voltage would have to be gigantic. These two observations are referred to as the virtual short-circuit principle[1], meaning that as long as the feedback is operating and the loop gain is very large, you can simply assume that the input voltage of the opamp is zero. This can make analyzing such circuits relatively easy. Consider the non-inverting amplifier shown in Figure 3. Recall from (6) that

$$V_{\mathrm{f}} = \frac{\mathrm{R2}}{\mathrm{R1} + \mathrm{R2}} V_{\mathrm{o}} \tag{16}$$

and from the virtual short circuit principle $V_{\mathrm{f}} = V_{\mathrm{i}}$, and so

$$V_{\mathrm{i}} = \frac{\mathrm{R2}}{\mathrm{R1} + \mathrm{R2}} V_{\mathrm{o}}\text{, or} \tag{17}$$

$$V_{\mathrm{o}} = \frac{\mathrm{R1} + \mathrm{R2}}{\mathrm{R2}} V_{\mathrm{i}}\,. \tag{18}$$

4.1 When the Virtual Short-Circuit Principle Does Not Apply

One quick way to determine whether a high-gain feedback loop is working properly is to examine the error signal V_{e}. If the feedback is working properly it should be very close to zero. In an opamp circuit, the error signal is the voltage between the two inputs. Most opamps suffer from a slight imbalance in their input stages that causes a small offset in the input, and so the voltage on the two inputs will never be precisely 0, but it should be no more than a few millivolts, 20mV at the most. Furthermore this offset voltage due to the imbalance would be constant, and so the error signal might have a constant offset component, but its value would vary little with changes in the input signal. But again, this assumes the feedback is working properly. If the feedback is broken, then you will generally see a voltage difference much greater than a few millivolts at the inputs of the opamp.

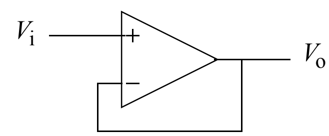
What would cause the loop to be broken? Well, the virtual short-circuit principle only applies to negative feedback. If the feedback loop uses positive feedback, either intensionally as in a latch, or unintentionally because the circuit was wired incorrectly, then the error signal will not be zero. Another very common situation that results in the loop being broken is if the expected output signal is outside the range that the opamp is capable of producing. All real opamps are provided power by applying voltages to their power supply pins. The opamp is incapable of driving its output to a voltage that is greater than the positive supply or less than the negative supply (often the negative supply is ground). For example, if an non-inverting amplifier has a gain of 10 and an input voltage of 1V, but the output of the opamp can produce a maximum of 5V, then feedback voltage will be 500mV (because from (3) f will be 0.1) and the error voltage will also be 500mV (because V_{i} = 1V and V_{f} = 500mV).

1. These two observations are also referred to as *the golden rules of opamps* by Horowitz and Hill. [2]

5 Voltage Follower

A special case of the non-inverting amplifier is shown in Figure 4. The virtual short circuit principle makes understanding this circuit trivial. From the virtual short circuit principle the voltage across the inputs of the opamp must be the same, since the negative input is connected to the output, this implies that the output voltage must exactly follow the input voltage. This circuit is a unity gain non-inverting amplifier.

FIGURE 4 *Voltage follower*



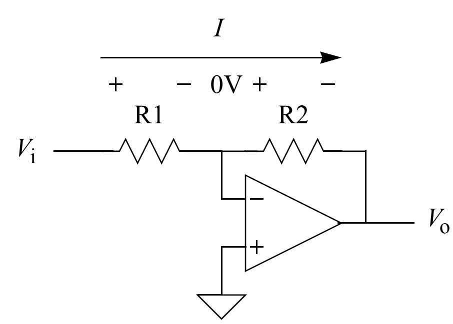
Why would you want such a circuit? Remember that the input current will be zero, so this circuit acts like a buffer, meaning that you can drive a heavy load (one that requires a large current) without loading the input.

6 Inverting Amplifier

An inverting amplifier is shown in Figure 5. From the virtual short-circuit principle, the equation that describes this amplifier is very simple to derive. First we observe that the positive input of the opamp is connected to ground, so its voltage is 0. From the virtual short circuit principle the voltage at the negative input must also be 0. Thus, the voltage across R1 is V_i. The current through R1 must be V_i/R1. Since the negative input of the opamp draws no current, all of this current must flow into R2, therefore the output voltage is

$$V_o = -\mathrm{R2}I = -\frac{\mathrm{R2}}{\mathrm{R1}}V_i. \tag{19}$$

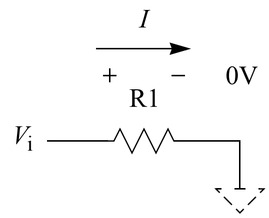
FIGURE 5 *Inverting amplifier*



The output voltage always has the opposite sign as the input voltage, and so this is an inverting amplifier.

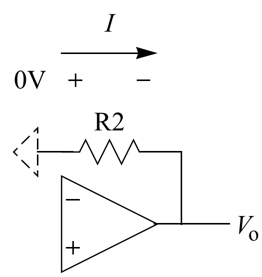
In this case the positive input is connected to ground, and the virtual short-circuit principle forces the voltage on the negative input to be 0. Thus, the negative input appears to be a virtual ground. This virtual ground allows the analysis of the circuit to be partitioned into two steps. First, you employ the virtual ground to analyze the portion of the circuit that precedes the virtual ground, as shown in Figure 6.

FIGURE 6 *Portion of the circuit that precedes the virtual ground*



From this we can see that $I = V_i/\mathrm{R1}$ by simply inspecting the circuit. The second step now involves analyzing the rest of the circuit, as shown in Figure 7.

FIGURE 7 *Portion of the circuit that follows the virtual ground*



From this we can see that $V_o = -\mathrm{R2}I$. Combining these two gives us

$$V_o = -\mathrm{R2}I = -\frac{\mathrm{R2}}{\mathrm{R1}}V_i. \tag{20}$$

7 Transresistance Amplifier

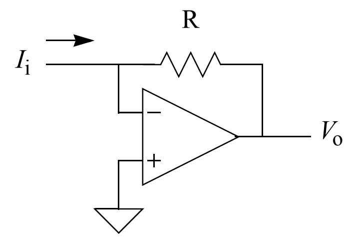
Consider the circuit shown in Figure 8. This circuit is just like the circuit of Figure 5 with the input resistor removed. Here we are assuming that the input is a current that is simply injected into the virtual ground. From (19) the output voltage is

$$V_o = -\mathrm{R}I. \tag{21}$$

Thus, this circuit acts as a current to voltage converter, which is referred to as a transresistance amplifier because its gain has units of resistance.

One can now think of the inverting amplifier as simply a voltage-to-current converter (a resistor, R1) feeding a current to voltage converter.

FIGURE 8 *Transresistance amplifier (a current to voltage converter)*



8 Integrator

Consider the circuit shown in Figure 9. This circuit is just like the circuit of Figure 8 with the feedback resistor replaced with a capacitor. The capacitor is described in Figure 10. The equation that describes a capacitor is:

$$I_C = C\frac{dV_C}{dt}. \tag{22}$$

The input pin of the amplifier absorbs no current and so the input current goes directly into the capacitor. As such, $I_i = I_C$. Because the input is a virtual ground, the voltage of the top plate of the capacitor is 0. The voltage of the bottom plate is $-V_o$, and so $V_C = -V_o$ and:

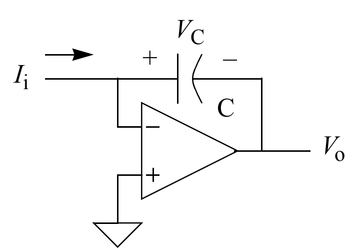
$$I_i = -C\frac{dV_o}{dt}, \tag{23}$$

which can be rewritten as

$$V_o = -\frac{1}{C}\int_t I_i d\tau. \tag{24}$$

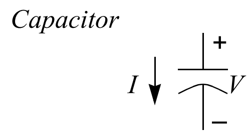
Thus, this circuit acts as a current integrator with a gain of 1/C.

FIGURE 9 *Current integrator*



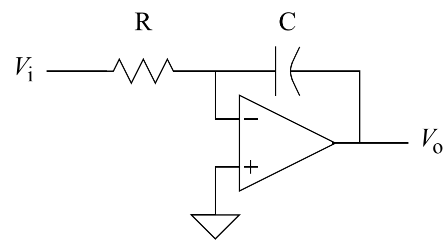
The input of this circuit is a virtual ground, so one can convert this circuit into a voltage integrator by simply adding a resistor to the input, as shown in Figure 11. Here the resistor is acting as a voltage to current converter, and then the integrator integrates the current.

FIGURE 10 *Capacitor*



$$I = C\frac{dV}{dt}$$

FIGURE 11 *Voltage integrator*

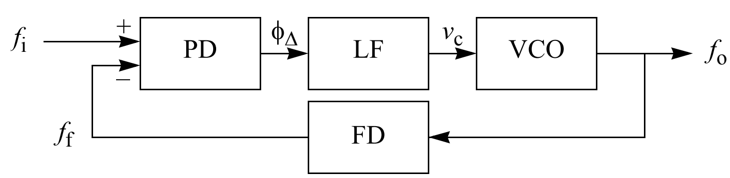


$$v_o = -\frac{1}{C}\int I_i dt = -\frac{1}{C}\int\frac{V_i}{R}dt = -\frac{i}{RC}\int V_i dt\,. \tag{25}$$

9 Phase-Locked Loop

Phase-locked loops are another example of negative feedback, except the signal being fed back is not a voltage or current as in the opamp circuits given previously, rather what is being fed back is a phase. Just like with the opamp circuits above, it is possible to implement a wide variety of functions with phase-locked loops. A phase-locked loop (PLL) operating as a frequency synthesizer shown in Figure 12. The synthesizer is constructed with a phase detector (PD), a loop filter (LF), a voltage-controlled oscillator (VCO), and a frequency divider (FD). Together, the PD, LF, and VCO play the role of the amplifier in the previous circuits, and the FD plays the role of the feedback.

FIGURE 12 *Block diagram of a phase-locked loop frequency synthesizer*

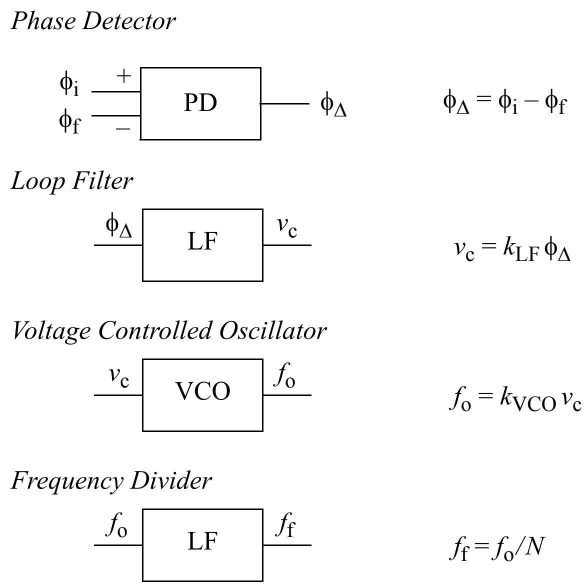


The virtual short-circuit principle can be used to understand the operation of this circuit. From the virtual short-circuit principle, as long as the feedback is operating properly, the two frequencies at the input of the phase detector must be the same. Therefore, if the frequency divider has a division ratio of N, then the output frequency must be N times higher than the input frequency.

$$f_o = Nf_f = Nf_i\,. \tag{26}$$

To derive this equation we needed the virtual short-circuit principle, which assumes that the feedback is operating as expected. The detailed equations for the PLL will now be derived to allow us to show the PLL does indeed work in this manner. To do so, the formulas that describe the behavior of the individual blocks are needed. They are summarized in Figure 13.

FIGURE 13 *Components that make up the frequency synthesizer.*



The input and output signals take the form:

$$v(t) = A\cos(\phi(t)) \tag{27}$$

where A is the amplitude of the signal and ϕ is the phase. Notice that the signal is not defined in terms of its frequency. Rather the phase is expected to be a monotonically increasing function of time with the frequency being the rate of change of the phase. The frequency of the signal can be computed from the phase with

$$f = \frac{d\phi(t)}{dt}. \tag{28}$$

In other words, frequency is the time derivative of phase, or phase is the time integral of frequency.

The amplitude of these signals are unimportant to the operation of the PLL and so will be largely ignored. The output of the phase detector is the phase difference between the signals at its two inputs. Thus,

$$\phi_\Delta = \phi_i - \phi_o. \tag{29}$$

The loop filter is required for rather subtle reasons involving the stability of the loop. Those reasons will not be further explained here and we will simply ignore the filtering aspect of the loop filter. However a function that is needed that has not yet been associated with any block is the need to convert the phase difference to a control signal that is capable of adjusting the frequency of the VCO. At a very minimum this function must change the units of the signal from radians to voltage, it may also include a gain term. So, let's associate this function to the loop filter. So, assume that the transfer function of the loop filter is:

$$v_{c} = k_{LF}\phi_{\Delta} . \tag{30}$$

The VCO is an oscillator whose frequency is controlled by an input voltage. The equation that describes a VCO is:

$$f_{o} = k_{VCO} v_{c} . \tag{31}$$

The frequency divider simply produces an output signal whose frequency is N times smaller than the signal at its input.

$$f_{f} = \frac{f_{o}}{N} . \tag{32}$$

It is the phase of the output of the divider that is fed back rather than its frequency, and from (28) the phase is

$$\phi_{f} = \int_{t} f_{f}\, d\tau . \tag{33}$$

Now the output frequency if the PLL can be derived as a function of the input frequency:

$$f_{o} = k_{VCO} k_{PD} \int_{t} \left(f_{i} - \frac{f_{o}}{N} \right) d\tau . \tag{34}$$

Finally, consider the behavior of the loop in steady state. This would be the behavior once it has settled down if the input frequency does not change. In this case, the output frequency would not change and the argument of the integral in (34) must be identically equal to zero, otherwise the integral would change with time and the output frequency would not be constant. This assumption of steady-state behavior implies that:

$$f_{i} - \frac{f_{o}}{N} = 0 , \tag{35}$$

which is nothing more than the virtual short-circuit principle applied to the loop. This can be rewritten as

$$f_{o} = N f_{i} .$$

Thus, the frequency synthesizer creates an output signal whose frequency is exactly N times larger than an input reference frequency. This output signal inherits the frequency precision and stability of the reference frequency. Furthermore, by changing N, you can change the output frequency. However, it is the nature of frequency dividers that N must be an integer, thus with this approach the output frequency is constrained to be an integer multiple of the reference frequency.

10 Conclusion

The ideas presented in this paper allow you to quickly gain a reasonable understanding of a wide variety of simply feedback circuits. When combined with a basic understanding of phasors[3] there is a surprising number of useful circuits you can quickly understand and design.

10.1 If You Have Questions

If you have questions about what you have just read, feel free to post them on the *Forum* section of *The Designer's Guide Community* website. Do so by going to *www.designers-guide.org/Forum*.

References

[1] Paul Gray, Paul Hurst, Stephen Lewis, and Robert Meyer. *Analysis and Design of Analog Integrated Circuits*. Fifth edition. Wiley. 2009.

[2] Paul Horowitz and Winfield Hill. *The Art of Electronics.* Second edition. Cambridge University Press. 1989.

[3] Ken Kundert. Introduction to Phasors. *www.designers-guide.org/Theory.*

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